The Magnetic Vector Potential

From the **magnetic** form of Gauss's Law $\nabla \cdot \mathbf{B}(\bar{r}) = 0$, it is evident that the **magnetic flux density** $\mathbf{B}(\bar{r})$ is a **solenoidal** vector field.

Recall that a solenoidal field is the **curl** of some other vector field, e.g.,:

$$\mathbf{B}(\bar{r}) = \nabla \times \mathbf{A}(\bar{r})$$

Q: *The magnetic flux density* $\mathbf{B}(\bar{r})$ *is the curl of* ***what*** *vector field ??*

A: The **magnetic vector potential** $\mathbf{A}(\bar{r})$!

The **curl** of the magnetic vector potential $\mathbf{A}(\bar{r})$ is **equal** to the magnetic flux density $\mathbf{B}(\bar{r})$:

$$\boxed{\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})}$$

where:

$$\text{magnetic vector potential } \doteq \mathbf{A}(\bar{\mathbf{r}}) \qquad \left[\frac{\text{Webers}}{\text{meter}}\right]$$

Vector field $\mathbf{A}(\bar{\mathbf{r}})$ is called the **magnetic** vector potential because of its **analogous** function to the **electric** scalar potential $V(\bar{\mathbf{r}})$.

An **electric** field can be determined by taking the **gradient** of the **electric potential**, just as the **magnetic** flux density can be determined by taking the **curl** of the **magnetic potential**:

$$\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{\mathbf{r}}) \qquad\qquad \mathbf{B}(\bar{\mathbf{r}}) = \nabla \times \mathbf{A}(\bar{\mathbf{r}})$$

Yikes! We have a **big problem**!

There are actually (infinitely) **many** vector fields $\mathbf{A}(\bar{\mathbf{r}})$ whose **curl** will equal an arbitrary magnetic flux density $\mathbf{B}(\bar{\mathbf{r}})$. In other words, given some vector field $\mathbf{B}(\bar{\mathbf{r}})$, the **solution** $\mathbf{A}(\bar{\mathbf{r}})$ to the **differential equation** $\nabla \times \mathbf{A}(\bar{\mathbf{r}}) = \mathbf{B}(\bar{\mathbf{r}})$ is **not unique** !

But of course, **we knew this**!

To **completely** (i.e., uniquely) specify a **vector** field, we need to specify **both** its divergence and its curl.

Well, we know the **curl** of the magnetic vector potential $\mathbf{A}(\bar{r})$ is equal to magnetic flux density $\mathbf{B}(\bar{r})$. But, what is the **divergence** of $\mathbf{A}(\bar{r})$ equal to ? I.E.,:

$$\nabla \cdot \mathbf{A}(\bar{r}) = \; ???$$

By answering this question, we are essentially **defining** $\mathbf{A}(\bar{r})$.

→ Let's define it in so that it makes our **computations easier**!

To accomplish this, we first start by writing **Ampere's Law** in terms of magnetic vector potential:

$$\nabla \text{x} \mathbf{B}(\bar{r}) = \nabla \text{x} \nabla \text{x} \mathbf{A}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

We recall from **section 2-6** that:

$$\nabla \text{x} \nabla \text{x} \mathbf{A}(\bar{r}) = \nabla\left(\nabla \cdot \mathbf{A}(\bar{r})\right) - \nabla^2 \mathbf{A}(\bar{r})$$

Thus, we can **simplify** this statement if we decide that the **divergence** of the magnetic vector potential is **equal to zero**:

$$\boxed{\nabla \cdot \mathbf{A}(\bar{r}) = 0}$$

We call this the **gauge equation** for magnetic vector potential. Note the magnetic vector potential $\mathbf{A}(\bar{r})$ is therefore **also** a **solenoidal** vector field.

As a result of this gauge equation, we find:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A}(\bar{\mathbf{r}}) &= \nabla\left(\nabla \cdot \mathbf{A}(\bar{\mathbf{r}})\right) - \nabla^2 \mathbf{A}(\bar{\mathbf{r}}) \\ &= -\nabla^2 \mathbf{A}(\bar{\mathbf{r}}) \end{aligned}$$

And thus **Ampere's Law** becomes:

$$\nabla \times \mathbf{B}(\bar{\mathbf{r}}) = -\nabla^2 \mathbf{A}(\bar{\mathbf{r}}) = \mu_0 \mathbf{J}(\bar{\mathbf{r}})$$

Note the Laplacian operator ∇^2 is the **vector Laplacian**, as it operates on **vector** field $\mathbf{A}(\bar{\mathbf{r}})$.

Summarizing, we find the magnetostatic equations in terms of **magnetic vector potential** $\mathbf{A}(\bar{\mathbf{r}})$ are:

$$\nabla \times \mathbf{A}(\bar{\mathbf{r}}) = \mathbf{B}(\bar{\mathbf{r}})$$

$$\nabla^2 \mathbf{A}(\bar{\mathbf{r}}) = -\mu_0 \mathbf{J}(\bar{\mathbf{r}})$$

$$\nabla \cdot \mathbf{A}(\bar{\mathbf{r}}) = 0$$

Note that the **magnetic** form of Gauss's equation results in the equation $\nabla \cdot \nabla \times \mathbf{A}(\bar{\mathbf{r}}) = 0$. **Why** don't we include this equation in the above list?

Compare the magnetostatic equations using the magnetic vector potential $\mathbf{A}(\bar{\mathbf{r}})$ to the electrostatic equations using the electric scalar potential $V(\bar{\mathbf{r}})$:

$$-\nabla V(\bar{r}) = \mathbf{E}(\bar{\mathbf{r}})$$

$$\nabla^2 V(\bar{r}) = -\frac{\rho_v(\bar{\mathbf{r}})}{\varepsilon_0}$$

Hopefully, you see that the two potentials $\mathbf{A}(\bar{\mathbf{r}})$ and $V(\bar{\mathbf{r}})$ are in many ways **analogous**.

For example, we know that we can determine a static field $\mathbf{E}(\bar{\mathbf{r}})$ created by sources $\rho_v(\bar{\mathbf{r}})$ either **directly** (from Coulomb's Law), or **indirectly** by first finding **potential** $V(\bar{\mathbf{r}})$ and then taking its derivative (i.e., $\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{r})$).

Likewise, the magnetostatic equations above say that we can determine a static field $\mathbf{B}(\bar{\mathbf{r}})$ created by sources $\mathbf{J}(\bar{\mathbf{r}})$ either **directly**, or **indirectly** by first finding **potential** $\mathbf{A}(\bar{\mathbf{r}})$ and then taking its derivative (i.e.,$\nabla \times \mathbf{A}(\bar{\mathbf{r}}) = \mathbf{B}(\bar{\mathbf{r}})$).

$$\rho_v(\bar{r}) \quad \Rightarrow \quad V(\bar{r}) \quad \Rightarrow \quad \mathbf{E}(\bar{r})$$

$$\mathbf{J}(\bar{r}) \quad \Rightarrow \quad \mathbf{A}(\bar{r}) \quad \Rightarrow \quad \mathbf{B}(\bar{r})$$